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The Hilbert–Schmidt cohomology for the Poincaré group

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Abstract. We consider cocycles for the Poincaré group $\mathscr{P}_+^\uparrow$ satisfying $\psi(gk)=V_g\psi(k)V_g^{-1}+\psi(g)$, where V is an irreducible, strongly continuous representation of $\mathscr{P}_+^\uparrow$, and $\psi(g)\in B(\mathscr{H})_2$, the Hilbert–Schmidt operators on the representation space $\mathscr{H}$ of V. It is shown that ψ must be of the form $\psi(g)=V_gAV_g^{-1}-A$ with $A\in B(\mathscr{H})_2$. The method can be extended to analyse the cocycles of products of arbitrary irreducible representations of $\mathscr{P}_+^\uparrow$, providing neither representation belongs to the little group of vanishing four-momentum.

1. Introduction

We solve problems which have their basis in quantum field theory. Our setting is a separable, complex Hilbert space $\mathscr{H}$ upon which a connected Lie group G acts through an irreducible, strongly continuous unitary representation $\{V_g\colon g\in \mathrm{G}\}$. Kraus and Streater (1980) and Polley *et al* (1980) posed problems of unitary implementability of group actions, in certain representations of free and quasi-free systems, in terms of cocycles with values in the Hilbert–Schmidt operators on $\mathscr{H}$, which we denote by $B(\mathscr{H})_2$. More precisely, the conditions led to

(1) a projection P which is complex-linear in $\mathscr{H}$, such that $V_gPV_g^{-1}-P\in B(\mathscr{H})_2$ and

(2) a bounded, self-adjoint, anti-linear operator A on $\mathscr{H}$ such that $V_gAV_g^{-1}-A\in B(\mathscr{H})_2$.

Conditions (1) and (2) express the covariance under G of certain representations ofhhe canonical anti-commutation and commutation relations, respectively, which are given by a projection P (Kraus and Streater 1980) or by a bounded, self-adjoint, anti-linear operator A (Polley *et al* 1980). For the CAR, these include all linear canonical transforms of the Fock representation. For the CCR, the restriction to bounded A has to be removed in order to cover the general case. This is done in the Appendix.

Our main result is that, for $\mathrm{G}=\mathscr{P}_+^\uparrow$ (the Poincaré group), both P and A must be Hilbert–Schmidt operators for (1) and (2) to be satisfied. This means that any linear canonical transformation, leading us out of the Fock representation of the CAR or the CCR, automatically destroys Poincaré covariance.

One can show that the function $g\to V_gTV_g^{-1}-T$, where T is as defined in (1) or (2), is strongly Hilbert–Schmidt continuous, i.e. $\|V_gTV_g^{-1}-T\|_2\to 0$ as $g\to id_{\mathrm{G}}$, where $\|\cdot\|_2$ denotes the Hilbert–Schmidt norm. The proof of this is contained in the Appendix.

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If we have $\mathscr{H} = L^2(M; S; \mathrm{d}\mu)$ where M is a measure space upon which G acts transitively, $\mathrm{d}\mu$ is a G-invariant measure on M, and S is a Hilbert space, then we can rewrite the action $V_g H V_g^{-1}$ for $H \in B(\mathscr{H})_2$ as one of the following.

(1) $V_g \otimes \bar{V}_g$ on $L^2(M \times M; S \times S; \mathrm{d}\mu \otimes \mathrm{d}\mu)$ when H is linear, where $\bar{V}_g = CV_gC$, and C is a conjugation on $\mathscr{H}$, i.e. C is anti-linear and $C^2 = 1$.

(2) $V_g \otimes V_g$ on $L^2(M \times M; S \times S\colon \mathrm{d}\mu \otimes \mathrm{d}\mu)$ when H is anti-linear.

2. Cocycles

The function $g \to V_g T V_g^{-1} - T \in B(\mathscr{H})_2$ (we do not for the moment distinguish between the cases when T is linear or anti-linear) is an example of a cocycle with values in $B(\mathscr{H})_2$ for the action $V_g(\cdot)V_g^{-1}$. A cocycle for the action $V_g(\cdot)V_g^{-1}$ with values in $B(\mathscr{H})_2$ is a function $\psi: \mathrm{G} \to B(\mathscr{H})_2$ such that

$$\psi(gk) = V_g \psi(k) V_g^{-1} + \psi(g) \qquad \text{for } g, k \in \mathrm{G}.$$

A cocycle $\psi: \mathrm{G} \to B(\mathscr{H})_2$ is said to be a true coboundary if there is an operator $H \in B(\mathscr{H})_2$ such that

$$\psi(g) = V_g H V_g^{-1} - H \qquad \text{for each } g \in \mathrm{G}.$$

We define the equivalence of two cocycles, ψ_1 and ψ_2, as follows. ψ_1 and ψ_2 are cohomologous (or, equivalent; or, in the same cohomology class) if their difference is a true coboundary, i.e. if there is an operator $H \in B(\mathscr{H})_2$ such that for each $g \in \mathrm{G}$

$$\psi_1(g) - \psi_2(g) = V_g H V_g^{-1} - H.$$

Our problems, therefore, become amenable to the theory of cocycles for unitary actions of groups, with values in a Hilbert space. Using the transformations of the action (given in § 1) $V_g(\cdot)V_g^{-1}$ into either $V_g \otimes \bar{V}_g$ or $V_g \otimes V_g$, it is possible to show that the cohomology classes of the cocycle $V_g T V_g^{-1} - T$, where T is either a projection or an anti-linear operator, are in a one-to-one correspondence with certain cohomology classes of the cocycles:

(1) $V_g \otimes \bar{V}_g F - F \in L^2(M \times M; S \times S; \mathrm{d}\mu \otimes \mathrm{d}\mu)$ if T is a projection;

(2) $V_g \otimes V_g F - F \in L^2(M \times M; S \times S; \mathrm{d}\mu \otimes \mathrm{d}\mu)$ if T is an anti-linear operator.

In each case, F is a linear functional on the Hilbert space. Of course, the correspondence is, in general, only into, not onto.

We now specialise to the case where $\mathrm{G} = \mathscr{P}_+^\uparrow(3+1)$, the Poincaré group for $3+1$ space–time dimensions. Then $M = \mathbb{R}^3$, $S = \mathbb{C}^{2s+1}$, where s is such that $2s+1$ is a positive integer, and $\mathrm{d}\mu = \mathrm{d}^3p/(p^2+m^2)^{1/2}$, $m \geq 0$. V is then an irreducible representation of $\mathscr{P}_+^\uparrow(3+1)$ on $\mathscr{H}$. Our approach now becomes the task of showing that in each case we have

$$F \in L^2(\mathbb{R}^3 \times \mathbb{R}^3; \mathbb{C}^{2s+1} \times \mathbb{C}^{2s+1}; \mathrm{d}\mu \otimes \mathrm{d}\mu) = \mathscr{H}$$

and this proves that every cocycle considered must be a true coboundary. A by-product of the analysis is a method which proves the triviality of any cocycle of $\mathscr{P}_+^\uparrow(3+1)$, for a tensor product of any two irreducible representations of $\mathscr{P}_+^\uparrow(3+1)$, provided that neither of them corresponds to the case of vanishing four-momentum.

3. Cocycles for $V \otimes \bar{V}$ and $V \otimes V$

The tensor products $V \otimes \bar{V}$ and $V \otimes V$ are reducible. Therefore we must examine the consequences of this for the cocycles of $V \otimes \bar{V}$ and $V \otimes V$. We have the following result (Parthasarathy and Schmidt 1972).

Theorem 1. Suppose U is a continuous unitary representation of a connected Lie group G acting on a Hilbert space $\mathscr{H}$, and that there is a standard measure α on a Borel space Ω such that

$$U = \int_\Omega^\oplus U^\omega \, \mathrm{d}\alpha(\omega)$$

where each U^ω is an irreducible continuous representation of G on $\mathscr{H}^\omega$, and

$$\mathscr{H} = \int_\Omega^\oplus \mathscr{H}^\omega \, \mathrm{d}\alpha(\omega).$$

Then, if $\delta(g)$ is a continuous cocycle of G for U, with values in $\mathscr{H}$, $\delta(g)$ can be written as

$$\delta(g) = \int_\Omega^\oplus \delta(\omega, g) \, \mathrm{d}\alpha(\omega)$$

where $\delta(\omega, g)$ is a continuous cocycle of G for U^ω, for all $\omega \in Y$, where $Y \subseteq \Omega$ with $\alpha(\Omega \backslash Y) = 0$.

The tensor products $V \otimes \bar{V}$ and $V \otimes V$ have been fully analysed by Schaaf (1970). One has the following results.

$$V \otimes \bar{V} = \int_\Omega^\oplus V^\omega \, \mathrm{d}\alpha(\omega)$$

where each V^ω is an irreducible representation of $\mathscr{P}_+^\uparrow(3+1)$ which corresponds to the case of $m^2 < 0$ (i.e. the representations V^ω belong to the little group SO(2, 1) = the Lorentz group in 2 + 1 space–time dimensions). This is the case when the mass carried by V is $m \geqslant 0$.

The case of $V \otimes V$ can be dealt with in the same manner: if V carries a mass $m \geqslant 0$ then we obtain

$$V \otimes V = \int_\Omega^\oplus V^\omega \, \mathrm{d}\alpha(\omega).$$

In this case, each V^ω is an irreducible representation of $\mathscr{P}_+^\uparrow(3+1)$ which corresponds to the case of $m^2 > 0$ (i.e. the representations V belong to the stability group SO(3)).

We have stated that *each* V^ω is of the type $m^2 < 0$ for $V \otimes \bar{V}$ and of the type $m^2 > 0$ for $V \otimes V$. This is not strictly correct, as we should include, in each case, representations which carry mass equal to zero. However, these occur with α-measure zero, so we may neglect them, and our analysis is true with or without taking account of them: we choose not to mention them beyond this remark.

It was shown by Araki (1969/1970, theorem 7.3) that if the connected group G contains an abelian normal subgroup N, then any cocycle $\psi(g)$ may be written as

$$\psi(g) = U_g F - F + \psi_1(g)$$

where U_g is the continuous unitary representation of G, F is some vector, which may or may not lie in the Hilbert space, and $\psi_1(g)$ is a cocycle with values in the space of vectors which are invariant under the action of N in the given representation.

In our case, $\mathcal{P}_+^\uparrow(3+1)$ contains the abelian normal subgroup $\mathbb{R}^4$, and thus each cocycle is decomposed in the above form. Suppose that $\psi_1(g)$ is a cocycle with values in the translation-invariant subspace of the Hilbert space of $V\otimes\bar{V}$ or $V\otimes V$. Then, in each direct integral decomposition, $\psi_1(\omega, g)$ should be invariant under the action of $\mathbb{R}^4$ in the representation V^ω, for each $\omega\in\Omega$ (neglecting sets of measure zero). In either case of V^ω, there is no translation-invariant vector in the corresponding Hilbert space, apart from the zero vector. Hence, $\psi_1(\omega, g)=0$ for all $g\in$ G and for each $\omega\in\Omega$, and it follows that the cocycles of $V\otimes\bar{V}$ or $V\otimes V$ are of the form

$$V_g\otimes\bar{V}_gF-F \qquad \text{or} \qquad V_g\otimes V_gF-F$$

for each $g\in\mathcal{P}_+^\uparrow(3+1)$. Each F can be considered as a linear functional on a dense set of the Hilbert space $\mathcal{H}=L^2(\mathbb{R}^3\times\mathbb{R}^3;\mathbb{C}^{2s+1}\times\mathbb{C}^{2s+1};\mathrm{d}\mu\otimes\mathrm{d}\mu)$. We write

$$F=\int_\Omega^\oplus F^\omega\,\mathrm{d}\alpha(\omega)$$

by which we mean

$$(F,f)=\int_\Omega(F^\omega,f^\omega)\,\mathrm{d}\alpha(\omega)$$

for each vector f in the dense set of definition of F.

Using theorem 1, we have

$$V_g\otimes\bar{V}_gF-F=\int_\Omega^\oplus(V_g^\omega F^\omega-F^\omega)\,\mathrm{d}\alpha(\omega)$$

and

$$V_g\otimes V_gF-F=\int_\Omega^\oplus(V_g^\omega F^\omega-F^\omega)\,\mathrm{d}\alpha(\omega).$$

For the case of $V\otimes V$, each V^ω is a representation of $\mathcal{P}_+^\uparrow(3+1)$ with mass $m>0$, and this implies that each cocycle function F is already in the Hilbert space, i.e. $V^\omega F^\omega-F^\omega$ is a true coboundary. This is shown in theorem 3 of Basarab-Horwath *et al* (1979). Hence we have that

$$F=\int_\Omega^\oplus F^\omega\,\mathrm{d}\alpha(\omega) \qquad \text{with } F^\omega\in\mathcal{H}^\omega.$$

However, this does not yet establish that

$$\|F\|^2=\int_\Omega\|F^\omega\|^2\,\mathrm{d}\alpha(\omega)$$

is a convergent integral. We shall prove this convergence in the next section, together with the similar case of $V\otimes\bar{V}$.

Before proceeding to the analysis of convergence, let us remark that for each continuous cocycle $\psi(g)$ of $\mathcal{P}_+^\uparrow(3+1)$—or any connected Lie group G—we may choose a cocycle $\psi'(g)$ in the same cohomology class as $\psi(g)$, such that $\psi'(g)$ is analytic at the identity of G. This implies, for cocycles of the type $V_g\otimes V_gF-F$ or $V_g\otimes\bar{V}_gF-F$, that we may assume F to obey

$$XF \in \mathscr{H}$$

where X is any representative of any Lie algebra element in the Lie algebra of G. Moreover, for any such X of the Lie algebra, we denote by X^{ω} the corresponding representative in the representation V^{ω} occurring in the direct integral decomposition, and we then have

$$XF = \int_{\Omega}^{\oplus} X^{\omega}F^{\omega} \, \mathrm{d}\alpha(\omega)$$

and the integral

$$\int_{\Omega} \|X^{\omega}F^{\omega}\|^2 \, \mathrm{d}\alpha(\omega)$$

is convergent.

We may assume, as well as this useful property, that each F is invariant under the action of SO(3). This follows from the fact that any cocycle $\psi(g)$ for a connected Lie group G, which also contains a compact subgroup K, is cohomologous to one which vanishes on the compact subgroup K. $\psi(g)$ can be chosen to have both this property and the analyticity property. As we have remarked, this allows us to assume F to be invariant under the action of SO(3), and this, in turn, implies that each F^{ω} in the direct integral decomposition may be assumed invariant under the action of SO(3).

Having made these remarks, we are now able to solve our problems.

4. Estimates for cocycle functions

We call the objects F^{ω}, such that $V_g^{\omega}F^{\omega} - F^{\omega} \in \mathscr{H}^{\omega}$ for each $g \in \mathscr{P}_+^{\uparrow}(3+1)$, cocycle functions. In this section, we prove certain estimates for these functions, which then allow us to prove the convergence of the direct integrals. An important component in the proof is Redheffer's inequality (Redheffer 1970), which is

$$r\int_0^{\infty} |u|^2 K'(p) \, \mathrm{d}p \leq \int_0^{\infty} |u'|^2 p^{1+r} K(p) \, \mathrm{d}p$$

where n is the dimension of the space on which u is defined. u is a function of the vector $\boldsymbol{p}$, but depends only on $p = |\boldsymbol{p}|$. K is some function such that $K'(p)$ is locally integrable.

Proposition 1. Suppose that F is a cocycle function for the irreducible representation V of $\mathscr{P}_+^{\uparrow}(3+1)$, carrying mass $m > 0$ and spin equal to s, acting in the Hilbert space $\mathscr{H} = L^2(\mathbb{R}^3; \mathbb{C}^{2s+1}; \mathrm{d}^3\boldsymbol{p}/(\boldsymbol{p}^2 + m^2)^{1/2})$. Then

$$\|F\|^2 \leq \sum_{\beta=1}^{3} \|\mathscr{I}_{\beta}F\|^2$$

where $\mathscr{I}_i$ is the generator of Lorentz boosts in the βth direction.

Proof. The generators of the rotation group can be written (Beckers and Jaspers 1978) as

$$\boldsymbol{J} = -\mathrm{i}(\boldsymbol{p} \times \nabla) + S_3(p + p_3)^{-1}(p_1, p_2, p + p_3), \qquad p = |\boldsymbol{p}|,$$

where S_3 is the diagonal Hermitian spin operator corresponding to the spin about the third axis. S_3 has at most one zero eigenvalue. From the assumed rotation-invariance of F, we obtain

$$JF = 0$$

and, exploiting the properties of S_3, we can write F as

$$F(\boldsymbol{p}) = u(p)e_0$$

where e_0 is the zero eigenvector of S_3 in the spin space $\mathbb{C}^{2s+1}$, and $u(p)$ is a scalar function depending on $p = |\boldsymbol{p}|$.

In the same paper (Beckers and Jaspers 1978), the Lorentz boosts $\mathcal{J}_\beta$, $\beta = 1, 2, 3$, are given in vector form by

$$\mathcal{J} = \mathrm{i}p_0\nabla + p_0p^{-1}S_3(p+p_3)^{-1}(p_2, -p_1, 0) + mp^{-2}S_1(p_1p_2(p+p_3), p_2^2(p+p_3)^{-1} - p, p_2)$$
$$+ mp^{-2}S_2(p - p_1^2(p+p_3)^{-1}, -p_1p_2(p+p_3), -p_1)$$

where $p_0 = (\boldsymbol{p}^2 + m^2)^{1/2}$ with $m > 0$. Of course, we assume $\mathcal{J}_\beta F \in \mathcal{H}$ for $\beta = 1, 2, 3$. Using the decomposition of F in terms of u, we obtain

$$\hat{\boldsymbol{p}} \cdot \mathcal{J}F = \mathrm{i}p_0p^{-1}(\boldsymbol{p} \cdot \nabla u)e_0 \qquad \text{where } \hat{\boldsymbol{p}} = \boldsymbol{p}/p.$$

The right-hand side is equal to $(\mathrm{i}p_0\, \mathrm{d}u/\mathrm{d}p)e_0$. Using Schwarz's inequality, one can show that

$$\|\hat{\boldsymbol{p}} \cdot \mathcal{J}F\|^2 \leqslant \sum_{\beta=1}^{3} \|\mathcal{J}_\beta F\|^2.$$

We also have the following calculations.

$$\|F\|^2 = \int |F|^2\, \mathrm{d}^3\boldsymbol{p}/(\boldsymbol{p}^2 + m^2)^{1/2} = 4\pi \int_0^\infty |u|^2p^2\, \mathrm{d}p/(\boldsymbol{p}^2 + m^2)^{1/2}.$$

Using Redheffer's inequality, we see that the right-hand side is dominated by the integral

$$4\pi \int_0^\infty \frac{|u'|^2p_0^2p^2\, \mathrm{d}p}{p_0} = \int \frac{|p_0u'|^2\, \mathrm{d}^3\boldsymbol{p}}{(\boldsymbol{p}^2 + m^2)^{1/2}} = \left\|\mathrm{i}p_0\frac{\mathrm{d}u}{\mathrm{d}p}\right\|^2.$$

It now follows that

$$\|F\|^2 \leqslant \sum_{\beta=1}^{3} \|\mathcal{J}_\beta F\|^2$$

and this establishes the proposition. Note that we have used $K(p) = p_0$ for the function $K(p)$ in Redheffer's inequality.

Proposition 2. Suppose that F is a cocycle function for the irreducible representation V of $\mathcal{P}_+^\uparrow(3+1)$, corresponding to the little group SO(2, 1)—i.e. $m^2 < 0$. Then

$$\|F\|^2 \leqslant \sum_{\beta=1}^{3} \|\mathcal{J}_\beta F\|^2.$$

Proof. The Hilbert space of the irreducible representation is given as

$$\mathcal{H} = L^2(\mathbb{R}^3; M; \theta(|\boldsymbol{p}|^2 - \beta^2)(|\boldsymbol{p}|^2 - \beta^2)^{-1/2}\, \mathrm{d}^3\boldsymbol{p})$$

where M is a Hilbert space, $\beta^2 = -m^2$ and θ is the Heaviside distribution. We shall use Redheffer's inequality, and for the function $K(p)$ we choose

$$K(p) = \theta(|\boldsymbol{p}|^2 - \beta^2)(|\boldsymbol{p}|^2 - \beta^2)^{1/2}$$

and this gives us

$$\begin{aligned} K'(p) &= \theta(|\boldsymbol{p}|^2 - \beta^2)(|\boldsymbol{p}|^2 - \beta^2)^{-1/2} + 2|\boldsymbol{p}|(|\boldsymbol{p}|^2 - \beta^2)^{1/2}\,\delta(|\boldsymbol{p}|^2 - \beta^2) \\ &= \theta(|\boldsymbol{p}|^2 - \beta^2)(|\boldsymbol{p}|^2 - \beta^2)^{-1/2}. \end{aligned}$$

For this case of V, the generators of the rotations are given by (see Beckers and Jaspers 1978)

$$\boldsymbol{J} = -\mathrm{i}(\boldsymbol{p} \times \nabla) + S_3(p_1(p+p_3)^{-1}, p_2(p+p_3)^{-1}, 1).$$

Using the assumed rotation-invariance of the cocycle function F, we obtain, as in proposition 1, that we may write $F(\boldsymbol{p}) = u(|\boldsymbol{p}|)e_0$, where e_0 is the eigenvector of S_3 belonging to the eigenvalue zero.

The generators for the boosts are written as

$$\begin{aligned} \mathcal{J} = \mathrm{i}p_0\nabla + S_3p_0p(p+p_3)^{-1}(p_2, -p_1, 0) + mR_1p^{-2}(p - p_1^2(p+p_3)^{-1}, -p_1p_2(p+p_3), -p_1) \\ + mR_2p^{-2}(-p_1p_2(p+p_3)^{-1}, p - p_2^2(p+p_3), -p_2) \end{aligned}$$

where R_1 and R_2 are the representations of the boosts of SO(2, 1) in the space M. Another calculation shows that

$$\hat{\boldsymbol{p}} \cdot \mathcal{J}F(\boldsymbol{p}) = \mathrm{i}p_0(\mathrm{d}u/\mathrm{d}p)e_0 \qquad \text{where } \hat{\boldsymbol{p}} = \boldsymbol{p}/p.$$

The argument now proceeds as in proposition 1, so that we obtain the same desired result:

$$\|F\|^2 \leqslant \sum_{\beta=1}^{3} \|\mathcal{J}_\beta F\|^2.$$

This establishes the proposition.

A by-product of proposition 2 is that the irreducible, space-like representations of $\mathcal{P}_+^\uparrow(3+1)$ have only true coboundaries as cocycles. The analogous result for time-like representations of $\mathcal{P}_+^\uparrow(3+1)$ was shown in the article Basarab-Horwath *et al* (1979).

5. Convergence of integrals

We now turn to the equation

$$XF = \int_\Omega^\oplus X^\omega F^\omega \,\mathrm{d}\alpha(\omega)$$

given at the end of § 3. As remarked, this integral converges, i.e.

$$\|XF\|^2 = \int_\Omega \|X^\omega F^\omega\|^2 \,\mathrm{d}\alpha(\omega) < \infty$$

where X is some generator, for either the representation $V \otimes \bar{V}$ or $V \otimes V$, of a one-parameter group of $\mathcal{P}_+^\uparrow(3+1)$, and X^ω is the corresponding generator for V^ω, in the direct-integral decomposition. In particular, this is true for the generators of the boosts, i.e.

$$\int_\Omega \|\mathscr{J}_\beta^\omega F^\omega\|^2 \,\mathrm{d}\alpha(\omega) < \infty \qquad \text{for } \beta = 1, 2, 3.$$

$\mathscr{J}_\beta^\omega$ is the generator of the boosts in the β direction for the representation V^ω.

Using the reults of proposition 1 and proposition 2, we have

$$\|F^\omega\|^2 \leqslant \sum_{\beta=1}^{3} \|\mathscr{J}_\beta^\omega F^\omega\|^2 \qquad \text{(almost-everywhere } \alpha)$$

and this implies that

$$\int_\Omega \|F^\omega\|^2 \,\mathrm{d}\alpha(\omega) \leqslant \sum_{\beta=1}^{3} \int_\Omega \|\mathscr{J}_\beta^\omega F^\omega\|^2 \,\mathrm{d}\alpha(\omega) < \infty.$$

From this it follows that if F satisfies either $V_g \otimes \bar{V}_g F - F \in \mathscr{H}$ or $V_g \otimes V_g F - F \in \mathscr{H}$ where $\mathscr{H} = L^2(\mathbb{R}^3 \times \mathbb{R}^3; M \times M; \mathrm{d}\mu \otimes \mathrm{d}\mu)$ and $g \in \mathscr{P}_+^\uparrow(3+1)$, then we must have $F \in \mathscr{H}$.

It now follows, from our remarks about cocycles $\psi(g)$ for $\mathscr{P}_+^\uparrow(3+1)$, with values in the Hilbert space $\mathscr{H}$, for the representations $V \otimes \bar{V}$ or $V \otimes V$, that every such cocycle is a true coboundary. In particular, it follows that if P is a projection with

$$V_g P V_g^{-1} - P \in B(\mathscr{K})_2$$

then $P \in B(\mathscr{K})_2$, and if A is self-adjoint and anti-linear, with

$$V_g A V_g^{-1} - A \in B(\mathscr{K})_2,$$

then $A \in B(\mathscr{K})_2$.

We remark that our methods can be used to prove that cocycles of $\mathscr{P}_+^\uparrow(3+1)$ for the representation $V^1 \otimes V^2$ are all true coboundaries. Here, V^1 and V^2 are both irreducible unitary representations of $\mathscr{P}_+^\uparrow(3+1)$ which do not correspond to the case of vanishing four-momentum. This is because the direct-integral decomposition of $V^1 \otimes V^2$ only contains (up to measure zero) irreducible representations for $m^2 < 0$ and $m^2 > 0$. Therefore our estimates are appropriate.

6. An alternative proof of the basic inequality

In the previous sections, the inequality

$$\|u\|^2 \leqslant \sum_{\beta=1}^{3} \|\mathscr{J}_\beta u\|^2 \qquad (*)$$

was proved in a direct way, for rotationally invariant distributions u acting on representation spaces of $V \otimes \bar{V}$ and $V \otimes V$. In this section, we want to sketch a less direct proof of the inequality, which is, however, valid for non-rotationally invariant u as well. Indeed, (*) can be traced back to the fact that only the principal series representation of the Lorentz subgroup of $\mathscr{P}_+^\uparrow(3+1)$ is contained in $V \otimes \bar{V}$ and $V \otimes V$. This is known to be true for all irreducible representations V of $\mathscr{P}_+^\uparrow(3+1)$ with mass $m > 0$ (Joos 1962) and mass $m = 0$ with discrete spin (Zmuidzinas (1966) treats spin $s = 0$, but this is not difficult to generalise to spin $s \neq 0$). Furthermore, the complex conjugation operator transforms the principal series representation into one which is unitarily equivalent to it (see the explicit construction by Naimark (1964a)), so that $\bar{V}$

and V are unitarily equivalent when regarded as representations of the Lorentz subgroup. According to Naimark (1964b), the direct product of two representations of the principal series type for SL(2, $\mathbb{C}$) contains only the principal series. We now exploit the Casimir operator $\mathcal{J}^2 - \boldsymbol{J}^2$ of SL(2, $\mathbb{C}$). In a principal series representation, this operator acts as a real number $\lambda > 1$ (Naimark 1964a). Thus, in a direct integral of principal series representations, we have that

$$\mathcal{J}^2 \geqslant 1 + \boldsymbol{J}^2 \geqslant 1$$

and this is the desired inequality. However, it is not known whether this inequality will extend the method to products of arbitrary irreducible representations of $\mathcal{P}_+^\uparrow(3+1)$, as does the method using the estimates of propositions 1 and 2.

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Appendix

In a symplectically transformed Fock representation, the Weyl operators are given by $W(f) = W_F(Tf)$, W_F denoting the usual Fock Weyl operator. Poincaré transformations act in accordance with the rule $U_g W(f) U_g^* = W(V_g f)$. However, this definition runs into inconsistencies unless $V_g \mathcal{D}_T \subset \mathcal{D}_T$ (the domain of the operator T). Therefore, we must include this condition in the definition of Poincaré covariance.

It is known (Polley *et al* 1980) that Poincaré covariance implies implementability of the symplectic operator $V_g T V_g^* T^{-1}$, and that, without loss of generality, T can be chosen to be in standard form $T = \exp A$, where A is anti-linear, self-adjoint, and has pure point spectrum. Then $V_g T V_g^* = \exp(V_g A V_g^*)$. Since $V_g \mathcal{D}_T \subset \mathcal{D}_T$, we also have $V_g \mathcal{D}_A \subset \mathcal{D}_A$. This is all that is needed to prove the 'only if' part of theorem 1 of the above-mentioned paper—including the case of unbounded A. As a result, $A_g = V_g A V_g^{-1} - A \in B(\mathcal{H})_2$ is a necessary condition for the covariance of the CCR representation fixed by A. It is, however, not a sufficient condition, so that the correspondence of the cohomology of cocycles A_g with the equivalence of CCR representations breaks down if A is unbounded. This does not affect the main conclusions of this paper since we derive the result $A \in B(\mathcal{H})_2$ from the condition $A_g \in B(\mathcal{H})_2$ without using CCR representations.

We now show how to prove that, even if A is unbounded, the cocycle A_g is still strongly continuous in the Hilbert–Schmidt norm. We note that we may reduce the case to A being linear and self-adjoint, since if A is anti-linear one can choose a conjugation C so that $AC = CA$, and this makes AC self-adjoint and linear (see Polley *et al* 1980). One can use the spectral theorem to show that $(h; A_g f)$ is measurable in g, being the pointwise limit of measurable functions, where $f \in \mathcal{D}_A$ and $h \in \mathcal{H}$. Further, if $f \in \mathcal{H}$ then there exists a sequence $\{f_n\} \subset \mathcal{D}_A$ so that $\lim_{n\to\infty} f_n = f$, whence

$$(h; A_g f) = \lim_{n\to\infty} (h; A_g f_n)$$

so that for all $h, f \in \mathscr{H}$, the function $(h; A_g f)$ is measurable. Suppose now that $H \in B(\mathscr{H})_2$; then we have for the inner product in $B(\mathscr{H})_2$ of H with A_g

$$(H, A_g)_2 = \sum_{j,k=1}^{\infty} (He_j; A_g e_k)$$

where $\{e_k\}$ is an orthonormal basis in $\mathscr{H}$. It now follows that $g \mapsto (H, A_g)_2$ is measurable, being the pointwise limit of measurable functions. So $g \mapsto A_g$ is a $B(\mathscr{H})_2$ weakly measurable function. Now it follows from theorem 6.3 of Araki (1969/1970) that $g \mapsto A_g$ is a $B(\mathscr{H})_2$ strongly continuous cocycle. This proves the contention made in the introduction to this article, and our assumptions are justified.

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